1) $Z(p)$ must be a positive real function of p;
2) $m_1(p)m_2(p) - n_1(p)n_2(p) = C(p^2-1)^n$.

Condition 2 implies that both numerator and denominator are of degree n and it is readily argued that an impedance function formed by terminating a section of transmission line in an indeterminant impedance function will remain indeterminant. Furthermore if $Z(p)$ is normalized so that the coefficient of p^n in its denominator is unity then C equals the terminating resistance.

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Vector Formulations for the Field Equations in Anisotropic Waveguides*

In the following we will exhibit vector formulations for the equations determining the different components of the electromagnetic field in a source-free uniform waveguide. All results will be stated without proof. The derivations are given elsewhere.[1] The vector formulations given below are applicable to uniform waveguides containing anisotropic media restricted only by the requirement that the permittivity ($\boldsymbol{\varepsilon}$) and permeability ($\boldsymbol{\mu}$) dyadics be independent of the axial coordinate z. For uniform waveguides (with the indicated restriction on $\boldsymbol{\mu}$ and $\boldsymbol{\varepsilon}$) we consider solutions to the Maxwell equations which display characteristic time and z dependence of the form $\exp i(\kappa z - \omega t)$. This assumption permits us to eliminate the z and t dependence from the Maxwell equations and rewrite these as:

$$\begin{bmatrix} \omega\boldsymbol{\epsilon} & -\nabla_t\times\mathbf{1} - i\kappa\mathbf{z}_0\times\mathbf{1}_t \\ -\nabla_t\times\mathbf{1} - i\kappa\mathbf{z}_0\times\mathbf{1}_t & \omega\boldsymbol{\mu} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{E} \\ i\mathbf{H} \end{bmatrix} = 0. \quad (1)$$

Here, as in all the matrix equations which follow, dot product multiplication is to be understood for the products of dyadics and vectors. In (1), $\mathbf{E}$ and $\mathbf{H}$ are, respectively, the steady-state electric and magnetic fields; ∇_t is the transverse gradient operator; $\mathbf{z}_0$ is the unit vector in the axial direction; $\mathbf{1}$ is the unit dyadic; and $\mathbf{1}_t$ is the unit transverse dyadic:

$$\mathbf{1}_t = \mathbf{1} - \mathbf{1}_z = \mathbf{1} - \mathbf{z}_0\mathbf{z}_0. \quad (2)$$

It is well known that the transverse field components, $\mathbf{E}_t$ and $\mathbf{H}_t$, constitute the independent field components. To eliminate the dependent longitudinal components from (1) it is convenient to express, *e.g.*, the $\boldsymbol{\varepsilon}$ dyadic as

$$\boldsymbol{\varepsilon} \rightarrow \begin{bmatrix} \boldsymbol{\varepsilon}_t & \boldsymbol{\varepsilon}_{tz} \\ \boldsymbol{\varepsilon}_{zt} & \epsilon_z \end{bmatrix} \quad (3)$$

where $\boldsymbol{\varepsilon}_t$ is a transverse dyadic, $\boldsymbol{\varepsilon}_{tz}$ and $\boldsymbol{\varepsilon}_{zt}$ are vectors, and ϵ_z is a scalar; *i.e.*,

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_t + \epsilon_z\mathbf{1}_z + \mathbf{z}_0\boldsymbol{\varepsilon}_{zt} + \boldsymbol{\varepsilon}_{tz}\mathbf{z}_0. \quad (4)$$

A similar representation is chosen for the $\boldsymbol{\mu}$ dyadic. It can then be shown that the (independent) transverse field components satisfy the following pair of (coupled) second-order differential equations (transverse vector eigenvalue problem):

$$\begin{bmatrix} \left(\omega\boldsymbol{\varepsilon}_t - \frac{1}{\omega}\nabla_t\times\mathbf{z}_0\frac{1}{\mu_z}\mathbf{z}_0\times\nabla_t - \frac{\omega}{\epsilon_x}\boldsymbol{\varepsilon}_{tz}\boldsymbol{\varepsilon}_{zt}\right) & \left(\frac{\boldsymbol{\varepsilon}_{tz}}{\epsilon_z}\mathbf{z}_0\times\nabla_t + \nabla_t\times\mathbf{z}_0\frac{\boldsymbol{\mu}_{zt}}{\mu_z} - i\kappa\mathbf{z}_0\times\mathbf{1}_t\right) \\ \left(\frac{\boldsymbol{\mu}_{tz}}{\mu_z}\mathbf{z}_0\times\nabla_t + \nabla_t\times\mathbf{z}_0\frac{\boldsymbol{\varepsilon}_{zt}}{\epsilon_z} - i\kappa\mathbf{z}_0\times\mathbf{1}_t\right) & \left(\omega\boldsymbol{\mu}_t - \frac{1}{\omega}\nabla_t\times\mathbf{z}_0\frac{1}{\epsilon_z}\mathbf{z}_0\times\nabla_t - \frac{\omega}{\mu_z}\boldsymbol{\mu}_{tz}\boldsymbol{\mu}_{zt}\right) \end{bmatrix} \begin{bmatrix} \mathbf{E}_t \\ i\mathbf{H}_t \end{bmatrix} = 0. \quad (5)$$

Once solutions to (5) are obtained, the corresponding longitudinal field components can be determined from a knowledge of the transverse components via

$$\begin{bmatrix} E_z \\ iH_z \end{bmatrix} = \begin{bmatrix} -\frac{1}{\epsilon_z}\boldsymbol{\varepsilon}_{zt} & \frac{1}{\omega\epsilon_z}\mathbf{z}_0\times\nabla_t \\ \frac{1}{\omega\mu_z}\mathbf{z}_0\times\nabla_t & -\frac{1}{\mu_z}\boldsymbol{\mu}_{zt} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{E}_t \\ i\mathbf{H}_t \end{bmatrix} \quad (6)$$

In general, to obtain solutions to the transverse vector eigenvalue problem (5) is a formidable task. We recall that even in the case of isotropic waveguides such solutions are usually obtained by replacing the vector eigenvalue problem by a pair of scalar eigenvalue problems whose eigenfunctions are (except in the case of TEM modes) proportional to the longitudinal field components. A similar technique may be employed in the general anisotropic situation under consideration here. It can be shown that the transverse field components are derivable from the longitudinal field components via

$$D(\kappa)\begin{bmatrix} \mathbf{E}_t \\ i\mathbf{H}_t \end{bmatrix} = \mathfrak{A}\mathfrak{B}\begin{bmatrix} E_z \\ iH_z \end{bmatrix} \quad (7)$$

where

$$D(\kappa) = \kappa^4 + \omega^2\kappa^2\,\mathrm{Tr}\,(\mathbf{z}_0\times\boldsymbol{\mu}_t\cdot\mathbf{z}_0\times\boldsymbol{\varepsilon}_t) + \omega^4\Delta_\epsilon\Delta_\mu, \quad (8)$$

$$\mathfrak{A} = k^2\Delta_\epsilon\Delta_\mu\begin{bmatrix} \omega\boldsymbol{\varepsilon}_t^{-1} & i\kappa\boldsymbol{\varepsilon}_t^{-1}\cdot\mathbf{z}_0\times\boldsymbol{\mu}_t^{-1} \\ i\kappa\boldsymbol{\mu}_t^{-1}\cdot\mathbf{z}_0\times\boldsymbol{\varepsilon}_t^{-1} & \omega\boldsymbol{\mu}_t^{-1} \end{bmatrix} + \kappa^2\begin{bmatrix} \omega\mathbf{z}_0\times\boldsymbol{\mu}_t\times\mathbf{z}_0 & -i\kappa\mathbf{z}_0\times\mathbf{1}_t \\ -i\kappa\mathbf{z}_0\times\mathbf{1}_t & \omega\mathbf{z}_0\times\boldsymbol{\varepsilon}_t\times\mathbf{z}_0 \end{bmatrix}, \quad (9)$$

$$\mathfrak{B} = \begin{bmatrix} -\omega\boldsymbol{\varepsilon}_{tz} & \nabla_t\times\mathbf{z}_0 \\ \nabla_t\times\mathbf{z}_0 & -\omega\boldsymbol{\mu}_{tz} \end{bmatrix}, \quad (10)$$

Δ_ϵ and Δ_μ are the determinants of (the matrix representations of) the $\boldsymbol{\varepsilon}_t$ and $\boldsymbol{\mu}_t$ dyadics, respectively, and $\mathrm{Tr}\,(\mathbf{z}_0\times\boldsymbol{\mu}_t\cdot\mathbf{z}_0\times\boldsymbol{\varepsilon}_t)$ is the trace of (the matrix representation for) the dyadic $\mathbf{z}_0\times\boldsymbol{\mu}_t\cdot\mathbf{z}_0\times\boldsymbol{\varepsilon}_t$. Further, it can be shown that the longitudinal field components satisfy the following pair of (coupled) second-order differential equations (scalar eigenvalue problem):

$$\begin{bmatrix} \epsilon_z E_z \\ i\mu_z H_z \end{bmatrix} = \widehat{\mathfrak{B}}\frac{\mathfrak{A}}{D(\kappa)}\mathfrak{B}\begin{bmatrix} E_z \\ iH_z \end{bmatrix} \quad (11)$$

where $D(\kappa)$, $\mathfrak{A}$, $\mathfrak{B}$ are defined in (7)–(9) and:

$$\widehat{\mathfrak{B}} = \begin{bmatrix} -\omega\boldsymbol{\varepsilon}_{zt} & \mathbf{z}_0\times\nabla_t \\ \mathbf{z}_0\times\nabla_t & -\omega\boldsymbol{\mu}_{zt} \end{bmatrix}. \quad (12)$$

Note that, in general, $1/D(\kappa)$ does not commute with either $\mathfrak{B}$ or $\widehat{\mathfrak{B}}$ since these contain differentiation operations. The reader may verify that the result in (11) reduces to the equation given by Kales[2] for the special case of an axially magnetized gyromagnetic medium (*i.e.*, where ϵ is a scalar and $\boldsymbol{\mu}_{tz} = \boldsymbol{\mu}_{zt} = 0$).

Any solution E_x, H_z to (11) yields, via (7), an eigenfunction (mode) of the transverse vector eigenvalue problem (5). This procedure is manifestly not valid when $D(\kappa) = 0$. Therefore, the set of vector eigenfunctions obtained from all the solutions to (11) becomes complete only when we add such vector eigenfunctions of (5) which are admitted when $D(\kappa) = 0$. That these additional eigenfunctions are the analogs of the TEM modes in the anisotropic case is evident from the fact that $D(\kappa) = (\omega^2\mu\epsilon - \kappa^2)^2$ for an isotropic medium with scalar μ and ϵ. The analogy to TEM modes indicated here should not be taken to imply any TEM-like properties of these eigenfunctions in the anisotropic case.

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[1] A. D. Bresler, "Vector Formulations for the Electromagnetic Field Equations in Uniform Waveguides Containing Anisotropic Media," Microwave Res. Inst., Polytechnic Inst. of Brooklyn, Brooklyn, N. Y., Rep. R-676-58; September, 1958.

[2] M. L. Kales, "Modes in waveguides that contain ferrites," *J. Appl. Phys.*, vol. 24, pp. 604–608; May, 1953.

An Extension of the Reflection Coefficient Chart to Include Active Networks*

INTRODUCTION

At a single frequency, a two-port can be represented by the scattering matrix [1], [5]

$$[b] = [S][a] \quad (1a)$$

$$b_1 = s_{11}a_1 + s_{12}a_2 \quad (1b)$$

$$b_1 = s_{21}a_1 + s_{22}a_2 \quad (1c)$$

where $s_{12} = s_{21}$ in the reciprocal two-port. If one defines an input reflection coefficient $\Gamma_{in} = b_1/a_1$ and a load reflection coefficient $\Gamma_L = a_2/b_2$ one can form

$$\Gamma_{in} = \frac{(s_{12}^2 - s_{11}s_{22})\Gamma_L + s_{11}}{1 - s_{22}\Gamma_L}. \quad (2)$$

Eq. (2) can be considered as a mapping of the Γ_L plane into the Γ_{in} plane. Since this is a bilinear transformation, angles between

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